Effective plasma model for the level correlations at the mobility edge



1995 J. Phys. A: Math. Gen. 28 3623

(http://iopscience.iop.org/0305-4470/28/13/008)

J. Phys. A: Math. Gen. 28 (1995) 3623–3640. Printed in the UK

# Effective plasma model for the level correlations at the mobility edge

V E Kravtsov† and I V Lerner‡

† International Centre for Theoretical Physics, PO Box 586, 34100 Trieste, Italy
and Institute of Spectroscopy, Russian Academy of Sciences, 142092 Troitsk, Moscow r-n, Russia
‡ School of Physics and Space Research, University of Birmingham, Birmingham B15 2TT, UK

Received 15 March 1995

**Abstract.** We consider the mapping of the energy level statistics for a $d$-dimensional disordered electron system at the mobility edge between metallic and insulating phases onto the model of a classical one-dimensional 'plasma' of fictitious particles. We deduce the effective pairwise interaction in the plasma that is consistent with the known *universal* two-level correlation function at the mobility edge and show that for level separation $\varepsilon \gg \Delta$ it decreases as $(\Delta/\varepsilon)^{\gamma}$ where $\Delta$ is the mean-level spacing, and $\gamma$ is the critical exponent related to the known critical exponent $\nu$ of the correlation length as $\gamma = 1-(\nu d)^{-1}$. We apply the plasma model to generalize Wigner's semicircle law, and to derive the large-energy asymptotic form of the nearest-level distribution. In the limit $\gamma \to 0$, which corresponds to the original Dyson mapping onto the plasma with logarithmic repulsion, we recover the classical results of Wigner–Dyson random matrix theory.

## 1. Introduction

Random matrix theory (RMT) has been suggested by Wigner [1] and Dyson [2] as a tool in creating a statistical description of quantized energy levels in complex and chaotic systems (see [3] for reviews). Gorkov and Eliashberg have conjectured [4] that the Wigner–Dyson statistics based on RMT are also capable of describing electron energy levels in a small isolated metallic particle at low temperatures. In this case, it is the presence of disorder that requires a statistical description of the levels. In considerations of disordered systems, one can use powerful analytical methods such as the impurity diagrammatic technique [5], and the nonlinear $\sigma$ model [6]. Thus, the conjecture of applicability of the Wigner–Dyson statistics to a quantum disordered electron system in the *metallic* phase has been proven analytically by Efetov [7] and Altshuler and Shklovskii [8]. Furthermore, an important universal extension of these statistics to the parametric regime (when level positions depend on some external parameter) has been constructed analytically for the disordered systems and generalized for chaotic systems [9]. A relation between the field-theoretical methods of [5–9] and the method of periodic orbits used in chaotic systems [10, 11] has recently been established by Argaman *et al* [12].

The Wigner–Dyson statistics do not depend on a spatial dimensionality, $d$. In contrast to that, properties of the disordered electron systems are strongly dimensionally dependent. Therefore, these statistics could be applicable to the disordered electron systems only in a specific regime. Namely, this is a 'zero-dimensional' ergodic regime where the dimensionality enters the results only via the mean-level spacing $\Delta$ setting the energy scale. The 'zero-dimensional' regime corresponds to a homogeneous distribution of the density of an excess particle over the whole sample. Only the diffusion mode with a zero wavevector $q = 0$ contributes to this distribution which is formed at times $t \gg \tau_D$. Here the ergodic time, $\tau_D = L^2/D$, is a typical time of the electron diffusion (with the diffusion coefficient $D$) through a sample of the size $L$. This sets a natural energy scale (the Thouless energy [13])

$$E_c \equiv \frac{\hbar}{\tau_D} = \frac{\hbar D}{L^2} \tag{1}$$

which restricts the applicability of the RMT approach to the energies $\varepsilon \ll E_c$.

The larger energies, $\varepsilon \gtrsim E_c$, or the shorter times, $t \lesssim \tau_D$, correspond to the non-ergodic diffusive regime. In this case, the excess-particle distribution is contributed from all the diffusion modes and thus depends essentially on the sample dimensionality. The level statistics in this regime have been shown by Altshuler and Shklovskii [8] to be totally different from the Wigner–Dyson statistics of RMT. However, the number of levels in the energy window of width $E_c$, $N(E_c) = E_c/\Delta = g$ ($g$ is the conductance measured in units of $e^2/\hbar$), diverges in the thermodynamic limit $L \to \infty$ for $d > 2$. Therefore, in the limit $L \to \infty$, the non-ergodic regime with $\varepsilon > E_c$ is unreachable in the metallic phase for any energy interval containing a finite number of levels, and the level statistics in metals remain zero-dimensional in the thermodynamic limit.

The situation is totally different in the vicinity of the Anderson metal–insulator transition that depends crucially on the spatial dimensionality [14]. The criterion for the transition at $d > 2$ is that the dimensionless conductance $g$ is of order 1. Therefore, the number of levels, $N(E_c)$, within the 'ergodic' energy window is of order 1, and the region of applicability of the zero-dimensional RMT description could include no more than a few levels. Energy windows that contain many levels are much wider than $E_c$ so that they correspond to the non-ergodic regime, and statistics to describe them are bound to be different from the Wigner–Dyson ones. On the other hand, the electron states at the transition point (mobility edge) are still delocalized, albeit quite different from the homogeneously extended states in the metallic phase. Thus the statistics should also be different from the Poisson limit applicable to the description of the energy levels in the insulating phase where localized electron states corresponding to different energies are not overlapping in space so that the energy levels are uncorrelated. As in the thermodynamic limit there is no relevant energy scale, apart from $\Delta$, in the vicinity of the transition†, the level statistics should be universal in contrast to those in the non-ergodic diffusive regime in the metallic phase.

The existence of universal statistics at the mobility edge has been *conjectured* by Shklovskii *et al* [15] who suggested that the nearest-level spacing distribution, $P(s)$, is a universal 'hybrid' of the Wigner–Dyson distribution at $s \lesssim 1$ and the Poisson distribution at $s \gtrsim 1$ ($s$ is a distance between the levels measured in units of the mean-level spacing $\Delta$). Had it been true, such an asymptotic behaviour would correspond to the absence of the level correlations for $s \gg 1$, as in the Poisson statistics applicable in the insulating regime. At the mobility edge, however, the spatial overlapping of the states should lead to level correlations. Indeed, it has been *analytically proved* [16] that the two-level correlation function has the following asymptotic behaviour:

$$R(s, s') = -c_{d\beta}\beta^{-1}|s - s'|^{-2+\gamma} \qquad |s - s'| \gg 1 \tag{2a}$$

† Here we do not consider the ballistic energy scale, $\varepsilon \gtrsim \hbar/\tau \sim \varepsilon_F$, as the mean number of levels at this scale, $\hbar/\tau\Delta$, diverges in the thermodynamic limit. For a finite system, the energy level statistics at the appropriate time-scale, $t \lesssim \tau$, should be insensitive to the Anderson transition.

$$\gamma = 1 - (\nu d)^{-1} . \tag{2b}$$

Here $c_{d\beta}$ is a certain positive number, $\beta$ is determined by the Dyson symmetry class, $\beta = 1, 2$, or 4 for unitary, orthogonal, and symplectic ensembles, respectively [3], $\gamma$ is a universal critical exponent which is related to the critical exponent $\nu$ of the correlation length diverging at the mobility edge [14], and the two-level correlation function is defined as

$$R(s, s') = \frac{1}{\rho^2}\langle \rho(\varepsilon)\,\rho(\varepsilon')\rangle - 1 \qquad s \equiv \varepsilon/\Delta \tag{3}$$

where $\rho(\varepsilon)$ is the exact one-electron density of states (DOS) including spin for a particular realization of disorder, $\langle\cdots\rangle$ denotes averaging over all the realizations, $\rho = \langle\rho(\varepsilon)\rangle$ is the average DOS which is energy-independent for $\varepsilon \gtrsim \hbar/\tau$ and $|\varepsilon - \varepsilon'| \ll \hbar/\tau$. The asymptotic behaviour (2) has been obtained in [16] by calculating all the diagrams (with accuracy up to a numerical coefficient) which turned out to be possible after taking into account the analytical properties of the diffusion propagator and certain scaling relations at the mobility edge.

Since the level correlations (2) are totally different from those in the Wigner–Dyson statistics (where $\gamma = 0$) and in the Poisson statistics (where they are absent at $s \neq 0$), the universal level statistics at the mobility edge should be drastically different from both Wigner–Dyson and the Poisson limit.

In this paper we suggest the mapping of the new universal level statistics onto an effective 'plasma' model, that can be used as a tool for exploring the large $s$ asymptotic properties. Such a way of studying the level statistics has been suggested by Dyson [17], who has shown that the joint eigenvalues' probability distribution in RMT could be exactly mapped to the Gibbs distribution for a classical one-dimensional plasma of fictitious particles with a repulsive logarithmic interaction. The eigenstates of the random matrices correspond to the particles in the plasma. The asymptotic properties of the plasma model, and thus the level statistics for large-level separations, can be determined within the conventional mean-field approach [3]. As well as RMT, the Dyson plasma model is applicable to the disordered electron systems in the ergodic regime.

The spectral distribution in the non-ergodic diffusive regime ($\varepsilon \gg E_c$) in the metallic phase, found by Altshuler and Shklovskii [8], may also be mapped onto an effective plasma model as has been recently suggested by Jalabert *et al* [18]. In this case, the effective plasma model was characterized with an *attractive* power-law tail in the interaction of the fictitious particles which reflected the level *attraction* at larger energy scales ($\gg E_c$) in the metallic phase noticed in the original paper of Altshuler and Shklovskii [8]. The joint probability distribution is unknown outside the ergodic regime so that the mapping is not exact. Its verification is in the fact that it reproduces the two-level correlation function found microscopically [8]. Then it could be possible to use such a model to determine different statistical properties in the non-ergodic regime. An important limitation, however, is that in the metallic phase the region of applicability for $d$-dependent non-ergodic statistics vanishes in the thermodynamic limit.

Here we develop the plasma model that corresponds to the level statistics at the mobility edge which is universal in the thermodynamic limit. We show that in this case, fictitious particles interact via a *repulsive* power law, so that the level *repulsion* persists at the mobility edge at any energy scale (see the footnote on the previous page). We then apply the effective plasma model for finding a shape of the average density of states (i.e. a generalization of Wigner's semicircle law [3]) and the nearest-level distribution $P(s)$. The latter result has been partly reported in [19]. The model can be used for further studies of statistical properties of the electrons at the mobility edge. We hope that it could be of interest by itself as a toy model which represents non-trivial *universal* level statistics different from both the Wigner–Dyson limit (particles with the logarithmic interaction) and the Poisson limit (non-interacting particles).

## 2. Dyson integral equation for the level density

We begin with outlining the Dyson scheme of mapping the level probability distribution onto the classical plasma model. Such a mapping is exact but it turns out to be particularly useful in the continuous limit of the model, when requirements necessary for the exact mapping may be relaxed. Such a relaxed way of deriving the plasma model, when one requires it to reproduce only the asymptotic form of the two-level correlation function rather than the whole distribution, will be extended here to the statistics at the mobility edge.

In RMT, the joint probability density of the eigenvalues for the ensembles of random Hermitian matrices [3] is given by

$$P(\{s_n\}) = \prod_{1\leqslant i\leqslant j\leqslant \mathcal{N}} |s_i - s_j|^\beta \exp\left(-\frac{\beta}{2}\sum_{j=1}^{\mathcal{N}} s_j^2\right) \qquad s \equiv \frac{\varepsilon}{\Delta} \tag{4}$$

where $\beta = 1, 2$ or 4 according as the ensemble is Gaussian orthogonal, unitary, or symplectic, and $\mathcal{N}$ is the matrix rank. One may represent this probability distribution in the form

$$P(\{s_n\}) = Z^{-1}\,\mathrm{e}^{-\beta W} \tag{5a}$$

$$W(\{s_n\}) = \sum_i V(s_i) + \sum_{i<j} f(|s_i - s_j|) \tag{5b}$$

where

$$f(|s_i - s_j|) = \ln|s_i - s_j|^{-1} \qquad V(s_i) = s_i^2/2\,. \tag{6}$$

Equation (5) correspond to the Gibbs probability distribution of a classical one-dimensional gas of $\mathcal{N}$ charged fictitious particles with the repulsive pairwise interaction $f$ in the presence of the confinement potential $V(s)$ which keeps the particles from escaping to infinity. With the harmonic confinement potential and the logarithmic interaction (equation (6)), the plasma model of (5), reproduces exactly the probability density of RMT (equation (4)).

The plasma model becomes simpler than the original RMT problem when one goes over to the continuous limit by introducing the particle density $\rho(s)$. Then, $W(\{s_n\}) \longrightarrow F[\rho(s)]$, and one arrives from (equation (5*b*) at the following free-energy functional:

$$F[\rho(s)] = \tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}s'\,\rho(s)\rho(s')f(|s-s'|) + \int_{-\infty}^{\infty}\mathrm{d}s\,\rho(s)V(s)\,. \tag{7}$$

The average particle density which corresponds to the average level density in the original model is convenient to write using the grand canonical variables:

$$\rho_0(s) \equiv \langle\rho(s)\rangle = Z^{-1}\int \rho(s)\,\mathrm{e}^{-\beta\Omega[\rho]}\,\mathcal{D}\rho \qquad Z = \int \mathrm{e}^{-\beta\Omega[\rho]}\,\mathcal{D}\rho \tag{8}$$

where $\Omega[\rho(s)] = F[\rho(s)] - \mu\mathcal{N}[\rho(s)]$ is the grand canonical potential, $\mathcal{N}[\rho(s)] = \int \mathrm{d}s\,\rho(s)$ is the particle number functional. In the mean-field (MF) approximation, when one neglects fluctuations $\rho(s) - \langle\rho(s)\rangle$ by using the saddle-point approximation in (8), $\rho_0(s)$ obeys the Dyson MF equation [17]:

$$\int_{-\infty}^{\infty}\mathrm{d}s'\,\rho_0(s')f(|s-s'|) = -V(s) + \mu \tag{9}$$

where the 'chemical potential' $\mu$ should be determined from the normalization condition

$$\int_{-\infty}^{\infty} \mathrm{d}s\, \rho_0(s) = \mathcal{N}\,. \tag{10}$$

In order to change properties of the plasma model, one can choose either $V$, or $f$ different from those in (6). It is convenient to consider an arbitrary potential $V(s)$ instead of the harmonic one. Then, it could be considered as the source term in (7), so that $\rho_0(s) = -\beta^{-1}\delta Z/\delta V(s)$. Further, taking the second functional derivative, one can express [20] the two-level correlation function (3), as follows:

$$R(s, s') = -\beta^{-1}\frac{\delta\rho_0(s)}{\delta V(s')}\,. \tag{11}$$

Applying the functional derivative (11), to (9) one arrives at the following equation:

$$\int_{-\infty}^{\infty} \mathrm{d}s'\, R(s', s'') f(|s - s'|) = -\beta^{-1}\delta\mu/\delta V(s'') + \beta^{-1}\delta(s - s'')\,. \tag{12}$$

The term with the variational derivative of the chemical potential $\mu$ is the only one that depends on the confinement potential $V(s)$ and thus is non-universal. The importance of this term depends on an asymptotic behaviour of $V(s)$ at large $s$.

For strong confinement when $V(s)$ increases faster than $|s|$, the average density $\rho_0(s)$ tends to a constant, and in the thermodynamic limit $\mathcal{N} \to \infty$ the correlation function $R^{\infty}(s', s'')$ only depends on the difference $|s' - s''|$. Thus the inhomogeneous term containing the variational derivative of the chemical potential vanishes in this limit and the corresponding limiting correlation function $R^{\infty}(s - s')$ does not depend on the choice of $V(s)$. Such a universality has also been proved [21] more rigourously, without using the MF approximation.

It is crucial for the above MF proof of the universality to require the quick increase of $V(s)$ for large $s$. In the opposite case of weak confinement [22], the average density $\rho_0(\varepsilon)$ is singular at the origin and decreases steeply with increasing $\varepsilon$. The correlation function $R^{\infty}(s, s')$ turns out to be not translationally invariant and different from Wigner–Dyson [22]. (In this case, one cannot neglect the variation of the chemical potential in (12) and the proof of the universality of $R^{\infty}(s, s')$ fails). However, the peculiar properties of the models with weak confinement, in particular a strong dependence of the averaged level density on energy, shows that such a model, being interesting by itself, could not be used for describing the level statistics in disordered electron systems where $\rho_0$ is constant in the whole energy region of interest. For such a description, one should consider only models with the strong confinement.

In this case, a change in the confinement potential has no impact on the correlation function, and the only way to construct an effective plasma model which corresponds to a different (from Wigner–Dyson) correlation function is to change the pairwise interaction $f$ in (5*b*). Then for a given correlation function $R$, equation (12) defines (in the thermodynamic limit when it is universal) the inverse problem: determining the effective interaction $f$ corresponding to $R$. In principle, it may be applied for finding $f$ in the non-ergodic regime, where the entire joint probability density similar to that in the ergodic regime, equation (4), is unknown. It has been done for the non-ergodic diffusive regime in metal [18] where the two-level correlation function is well known, $R \propto |s - s'|^{d/2-2}$ for $s \gtrsim g$ [8]. In that case the effective interaction for large energies, $s \gtrsim g \gg 1$ has been found as a power-law function with the exponent $d/2 > 1$.

The asymptotic behaviour of $R^{\infty}(s - s')$ at the mobility edge (2), is characterized by a faster decrease than in the diffusive regime in metal, this decrease taking place at the entire energy scale $|s - s'| \gtrsim 1$ (see the previous footnote). In such a case, the solution to (12) depends crucially on the behaviour of the correlation function $R^{\infty}(s - s')$ at $|s - s'| \lesssim 1$ which is unknown. An attempt to use the asymptotic form of (2) for all energies, including $|s - s'| \lesssim 1$, would lead to an ill-defined strongly singular integral equation for $f$.

Instead of considering (12), we will solve a generalized version of the Dyson equation (9), *assuming* a long-range power-law interaction, such as the integral $\int ds\, f(s)$ being divergent at large $s$. As the energies $|s - s'| \lesssim 1$ make no relevant contribution to the solution of such an equation, $\rho_0(s)$, at large $s$, we may consider the model with the long-range power-law interaction for all energies. As in the case of the standard Dyson model, one can apply the MF approach to this model. The appropriate weakly singular integral equation may be solved by the standard methods [23]. Having found this solution for an arbitrary confining potential $V(s)$, one can use (11) to obtain the large-$s$ behaviour of the correlation function and, therefore, to determine which plasma model is *consistent* with the known microscopic results (2).

It should be emphasized that considering the model with the long-range interaction, we are dealing with an incompressible liquid: in such a case the sum rule, $\int_{-\infty}^{\infty} ds' R(s, s') = 0$, which is valid for any finite system, also holds in the thermodynamic limit, $\mathcal{N} \to \infty$. This is inconsistent with the microscopic results for the level statistics at the mobility edge, where the sum rule is violated in this limit [24]. However, as we will describe elsewhere, the sum rule violation is due to some specific statistical properties at a small energy scale, while the effective plasma model suggested here fully explains the statistical properties that are determined by the level correlations at a large energy scale only.

Note that a non-logarithmic interaction leads to the disappearance of the invariant measure $\prod |s_i - s_j|^{\beta}$ in the distribution (4). It means that there is no straightforward generalization of the invariant matrix ensembles to describe the level statistics at the mobility edge. Notwithstanding this, an effective plasma model appears to be quite a convenient tool for finding asymptotic characteristics of these statistics.

## 3. Effective interaction

We consider the generalized Dyson MF equation for $\rho_0$,

$$\int_{-\mathcal{E}}^{\mathcal{E}} ds'\, \rho_0(s) f(|s - s'|) = -V(s) + \mu \tag{13a}$$

using the following ansatz for its kernel:

$$f(|s - s'|) = \frac{A}{|s - s'|^{\gamma}} \qquad 0 < \gamma < 1 . \tag{13b}$$

As well as in the Wigner–Dyson statistics, we envisage that for finite $\mathcal{N}$ the equilibrium density $\rho_0(s)$ is non-zero only on a finite support, $s \in [-\mathcal{E}, \mathcal{E}]$, with the band-edge $\mathcal{E}$ to be found from the normalization condition (10). After finding $\rho_0(s)$ as a functional of $V$ and using (11), we shall determine $R(s, s')$ which should be an even function of $s - s'$ in the limit $\mathcal{N} \to \infty$. Comparing this with the microscopic result (2$a$), we shall determine the coefficient $A$ and prove that the exponent $\gamma$ in (13$b$) is the same as that in (2$a$). Although the asymptotic behaviour (2$a$) is only valid for large $s$, this is also the region of validity of the continuous MF approach. So we can use the asymptotic expression (2$a$) for all $s$. Although a 'true' kernel of the integral equation (13$a$) should be different for $s \lesssim 1$, such a difference could only lead to small corrections for the equilibrium $\rho_0(s)$, and is totally irrelevant for the statistics at large $s$ that we intend to study.

Now we define new variables, $z = s/\mathcal{E}$, $t = s'/\mathcal{E}$, and

$$u(t) = A\mathcal{E}^{1-\gamma}\rho_0(t\mathcal{E}) \qquad v(z) = V(z\mathcal{E}) - \mu \tag{14}$$

thus reducing (13*a*) to

$$\int_{-1}^{1} \mathrm{d}t\, \frac{u(t)}{|t-z|^{\gamma}} = -v(z) \qquad z \in [-1; 1]. \tag{15}$$

This weakly singular integral equation ($0 < \gamma < 1$) is solved in appendix A, using the technique described in [23]. The solution in the original variables, as in (13), may be formally written down as

$$\rho_0(s) = A^{-1}\frac{\cos^2(\pi\gamma/2)}{\pi^2}\mathrm{B}\left(\gamma, \frac{1-\gamma}{2}\right)\left(1+\frac{s}{\mathcal{E}}\right)^{\frac{\gamma-1}{2}} \frac{\mathrm{d}}{\mathrm{d}s}\left\{\int_s^{\mathcal{E}} \mathrm{d}t\, \left(1+\frac{t}{\mathcal{E}}\right)^{1-\gamma} (t-s)^{\frac{\gamma-1}{2}}\right.$$
$$\left.\times\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{-\mathcal{E}}^{t} \mathrm{d}\tau\, \left(1+\frac{\tau}{\mathcal{E}}\right)^{\frac{\gamma-1}{2}} (t-\tau)^{\frac{\gamma-1}{2}}(V(\tau)-\mu)\right]\right\}. \tag{16}$$

To make the ansatz (13*b*) self-consistent, we first need to find the two-particle correlation function from (11) and then to compare it with the known one (2*a*). On performing a trivial integration over the $\delta$ function arising from the differentiation (11), we have

$$R(s,s') = B\frac{\mathrm{d}}{\mathrm{d}s}\left\{\int_s^{\mathcal{E}} \mathrm{d}t\, \left(1+\frac{t}{\mathcal{E}}\right)^{1-\gamma} (t-s)^{\frac{\gamma-1}{2}}\frac{\mathrm{d}}{\mathrm{d}t}\left[(t-s')^{\frac{\gamma-1}{2}}\theta(t-s')\right]\right\} \tag{17}$$

where $\theta(t-s')$ is the Heaviside step-function, and

$$B = -\frac{\cos^2(\pi\gamma/2)}{\beta\pi^2 A}\left(1+\frac{s}{\mathcal{E}}\right)^{\frac{\gamma-1}{2}}\left(1+\frac{s'}{\mathcal{E}}\right)^{\frac{\gamma-1}{2}}\mathrm{B}\left(\gamma, \frac{1-\gamma}{2}\right). \tag{18}$$

We are interested in the thermodynamic limit, when $\mathcal{N} \to \infty$, and thus the band-edge $\mathcal{E} \to \infty$. Then all further considerations will be made for the energy region

$$1 \ll s \ll \mathcal{E}. \tag{19}$$

This corresponds to the diffusive regime, $1 \ll \varepsilon/\Delta \ll \hbar/(\tau\Delta) \to \infty$, in the original problem of the energy levels in a disordered electron system. In this limit, the coefficient $B$ (18), becomes merely a number. Firstly, we calculate the integral for $s > s'$ when the $\theta$ function in (17) equals 1. We have

$$R(s,s') = \tfrac{1}{2}(\gamma-1)B\frac{\mathrm{d}}{\mathrm{d}s}\int_s^{\mathcal{E}} \mathrm{d}t\, \left(1+\frac{t}{\mathcal{E}}\right)^{1-\gamma}(t-s)^{\frac{\gamma-1}{2}}(t-s')^{\frac{\gamma-3}{2}}$$
$$= \tfrac{1}{2}(\gamma-1)B\int_s^{\mathcal{E}} \mathrm{d}t\, (t-s)^{\frac{\gamma-1}{2}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(1+\frac{t}{\mathcal{E}}\right)^{1-\gamma}(t-s')^{\frac{\gamma-3}{2}}\right].$$

In this integral one can take the limit $\mathcal{E} \to \infty$ to obtain

$$R(s,s') = \tfrac{1}{4}(\gamma-1)(\gamma-3)B\int_s^{\infty} \mathrm{d}t\, (t-s)^{\frac{\gamma-1}{2}}(t-s')^{\frac{\gamma-5}{2}} = \frac{\frac{1}{2}(\gamma-1)^2\mathrm{B}(\frac{\gamma-1}{2}, 1-\gamma)}{|s-s'|^{2-\gamma}}. \tag{20}$$

For $s < s'$ one integrates by parts in (17) thus reducing it to the same expression as in (20) with $s \leftrightarrows s'$. Then, substituting $B$ (equation (18)), after elementary transformations we obtain

$$R(s,s') = -\frac{(1-\gamma)\cot(\pi\gamma/2)}{2\pi A\beta}\frac{1}{|s-s'|^{2-\gamma}}. \tag{21}$$

Comparing this with the microscopic expression for the two-level correlation function (2*a*), we arrive finally at the following expression for the effective interaction of (13*b*):

$$f(|s-s'|)=\frac{A}{|s-s'|^{\gamma}}\qquad A=\frac{(1-\gamma)\cot(\pi\gamma/2)}{2\pi c_{d\beta}}. \tag{22}$$

Note that

$$\lim_{\gamma\to 0} f(|s-s'|)=\lim_{\gamma\to 0}\frac{1}{\pi^2 c_{d\beta}}\left[\frac{1}{\gamma}+\ln|s-s'|^{-1}\right]. \tag{23}$$

Thus, in the limit $\gamma\to 0$ and with the choice of the constant $c_{d\beta}=1/\pi^2$, the solution (16) should be valid for the standard Wigner–Dyson statistics. We will show later that this is indeed the case. Obviously, the main advantage of the representation of (16) is that it gives a solution for different statistics as well.

## 4. A generalization of the semicircle law

The first question we address is how the change of the interaction is reflected in the average density of states, $\rho_0(s)$. In contrast to the correlation function, $\rho_0(s)$ does depend on a choice of the confinement potential $V$. In the case of the logarithmic interaction and the harmonic confinement potential it describes by Wigner's semicircle law [3]. In order to generalize it to the non-logarithmic interaction (22), as well as for other applications, it is convenient to represent the solution to (13) in the form, explicitly symmetric for an even potential $V(s)$. To this end, note that for $V(s)=V(-s)$ one could limit the integration in (13*a*) to positive $s'$ only, by having substituted the kernel (22) with the following one:

$$f(|s-s'|)=\frac{(1-\gamma)\cot(\pi\gamma/2)}{2\pi c_{d\beta}}\left[\frac{1}{|s-s'|^{\gamma}}+\frac{1}{(s+s')^{\gamma}}\right]. \tag{24}$$

Then, using a method similar to that described in appendix A, one obtains the solution of the integral equation (13*a*) with the kernel (24) as follows:

$$\rho_0(s)=\frac{2^{2-\gamma}c_{d\beta}\sin(\pi\gamma)}{\pi(1-\gamma)}B\left(\gamma,\tfrac{1}{2}-\tfrac{1}{2}\gamma\right)\times\frac{\mathrm{d}}{\mathrm{d}(s^2)}\left\{\int_s^{\mathcal{E}}\mathrm{d}t\,t^{2-\gamma}(t^2-s^2)^{\frac{\gamma-1}{2}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t\mathrm{d}\tau\,(t^2-\tau^2)^{\frac{\gamma-1}{2}}(V(\tau)-\mu)\right]\right\}. \tag{25}$$

Equations (16) and (25) represent the same solution for any even function $V(\tau)$. The last one is more convenient for integration with any even $V(\tau)$. The integrals with $V(\tau)=\tau^{2n}$ are calculated in appendix B. Using these results, one obtains for the harmonic potential $V(\tau)=\tau^2/2$:

$$\rho_0(s)=\frac{2c_{d\beta}\sin\frac{\pi\gamma}{2}}{1-\gamma}\left\{\frac{(\mathcal{E}^2-s^2)^{\frac{1+\gamma}{2}}}{\gamma(1+\gamma)}-\left(\frac{\mathcal{E}^2}{2\gamma}-\mu\right)(\mathcal{E}^2-s^2)^{\frac{\gamma-1}{2}}\right\}.$$

The 'chemical potential', $\mu$ can be found by requiring $\rho_0$ to be non-singular near the band edge $\mathcal{E}$, and the value of $\mathcal{E}$ is related to the number of 'particles' $\mathcal{N}$ (i.e. to the number of the energy levels in the original problem) by the normalization condition (10). Thus we obtain

$$\rho_0(s)=\frac{2\sin\frac{\pi\gamma}{2}c_{d\beta}}{\gamma(1-\gamma^2)}(\mathcal{E}^2-s^2)^{\frac{1+\gamma}{2}}. \tag{26}$$

The limit $\gamma = 0$ corresponds to the logarithmic interaction. In this limit (26) goes over to the standard Wigner's semicircle, provided that the constant $c_{d\beta}$ is chosen to be equal to $1/\pi^2$, as in (23).

## 5. The nearest-level distribution

The distribution density, $P(s)$, describes the probability of finding the nearest adjacent level at the distance $s = \omega/\Delta$ from a given level. In the case of the Wigner–Dyson statistics, it is given by the 'Wigner surmise' [3]:

$$P(s) = B_1 s^\beta \exp\left(-B_2 s^2\right). \tag{27}$$

Here the constants $B_{1,2}$ may be found from the normalization, and from the first moment of $P(s)$ (which gives the average distance between the levels, $\langle s \rangle = 1$). This surmise is exact only for the case $\mathcal{N} = 2$ but quite accurately describes the nearest-level distribution for all $s$ even in the limit $\mathcal{N} \to \infty$. Equation (27) manifests the level repulsion in the Wigner–Dyson statistics: *the probability to find the nearest level at the distance $s$ vanishes* with $s \to 0$.

In general, the probability of finding the nearest level in a system of the long-range correlated levels is contributed by all the levels. However, $P(s)$ at $s \ll 1$ is governed by a single pair of levels only. It can be found (see, e.g. [8]) by diagonalizing a $2 \times 2$ matrix whose off-diagonal elements $V_{12}$ are real, complex, or quarternion numbers for $\beta = 1, 2,$ or 4, respectively. Thus, the level repulsion at $s \ll 1$, $P(s) \propto s^\beta$, should remain valid in the disordered electron system in any regime. On the contrary, the asymptotic behaviour of $P(s)$ at $s \gg 1$ depends crucially on a long-range character of the level–level correlations, and can be found within the plasma model [3].

In terms of the plasma model (5), $P(s)$ is proportional to the probability of finding a 'gap' (i.e. a region that contains no 'particles') of width $s$. In the MF approximation, this probability is obtained [3] from (5*a*) as

$$P(s) \propto \exp\left[-\beta(F_s - F_0)\right]. \tag{28}$$

Here $F_0$ is the free energy of the one-dimensional plasma with a homogeneous distribution given by (7), while $F_s$ is the free energy of the plasma which is distributed along the straight line with a gap $s$ around its centre:

$$F_s = \tfrac{1}{2} \int_{|x| \geqslant \frac{s}{2}} \mathrm{d}x \int_{|x'| \geqslant \frac{s}{2}} \mathrm{d}x' \, \rho_s(x) \rho_s(x') f(|x - x'|) + \int_{|x| \geqslant \frac{s}{2}} \mathrm{d}x \, \rho_s(x) V(x). \tag{29}$$

Here $\rho_s(x)$, the density of the distribution with the gap, obeys the following MF equation:

$$\int_{|x| \geqslant \frac{s}{2}} \mathrm{d}x \, \rho_s(x) f(|x - x'|) = -V(x') + \mu_s \tag{30}$$

where the appropriate chemical potential, $\mu_s$, should be determined from the normalization condition similar to that in (10):

$$\int_{|x| \geqslant \frac{s}{2}} \mathrm{d}x \, \rho_s(x) = \mathcal{N}. \tag{31}$$

Then we consider an asymptotic behaviour of $P(s)$ in the thermodynamic limit (19). In this limit, although both $F_0$ and $F_s$ depend on a choice of the confinement potential, their difference does not. Thus the nearest-level distribution (28), is $V$-independent, similar to the correlation function (11). To prove it, firstly we use the MF equation (30) and the normalization condition (31) to rid of explicit dependence of $F_s$ on $V$ in (29) as follows:

$$F_s = -\tfrac{1}{2}\int_{|x|\geqslant\frac{s}{2}} \mathrm{d}x \int_{|x'|\geqslant\frac{s}{2}} \mathrm{d}x'\, \rho_s(x)\rho_s(x')f(|x-x'|) + \mu_s\mathcal{N}\,. \tag{32}$$

A similar expression for $F_0$ follows from (7) and (9). Then

$$F_s - F_0 = -\tfrac{1}{2}\int_{|x|\geqslant\frac{s}{2}} \mathrm{d}x \int_{|x'|\geqslant\frac{s}{2}} \mathrm{d}x'\, \rho_s(x)\rho_s(x')f(|x-x'|)$$
$$+\tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}x'\, \rho_0(x)\rho_0(x')f(|x-x'|) + (\mu_s-\mu)\mathcal{N}\,. \tag{33}$$

The change in the 'chemical potential' due to the gap formation, $\mu_s - \mu$, is found from (9) and (30) as

$$\mu_s - \mu = \int_{|x'|\geqslant\frac{s}{2}} \mathrm{d}x'\, \rho_s(x)f(|x-x'|) - \int_{-\infty}^{\infty} \mathrm{d}x'\, \rho_0(x')f(|x-x'|)\,. \tag{34}$$

It is convenient to use the following expression for $\mathcal{N}$:

$$\mathcal{N} = \int_{|x|\geqslant\frac{s}{2}} \mathrm{d}x\, \rho_0(x)\,. \tag{35}$$

This is different from the exact expression (10) by a term of order $s$ which is negligible in the thermodynamic limit. Now, substituting (35) and (34) into (33), we obtain after straightforward transformations:

$$F_s - F_0 = -\tfrac{1}{2}\int_{|x|\geqslant\frac{s}{2}} \mathrm{d}x \int_{|x'|\geqslant\frac{s}{2}} \mathrm{d}x'\, \delta\rho_s(x)\delta\rho_s(x')f(|x-x'|)$$
$$+\tfrac{1}{2}\int_{|x|\leqslant\frac{s}{2}} \mathrm{d}x \int_{|x'|\leqslant\frac{s}{2}} \mathrm{d}x'\, \rho_0(x)\rho_0(x')f(|x-x'|) \tag{36}$$

where the change in the density due to the gap formation, $\delta\rho_s(x) \equiv \rho_s(x) - \rho_0(x)$, decreases rapidly for $x \gg s$. This means, in particular, that $\mu_s - \mu$ is of order $s/\mathcal{E}$ thus vanishing in the thermodynamic limit. Neglecting this change, we obtain from (34) the MF equation for $\delta\rho_s(x)$ as follows:

$$\int_{|x'|\geqslant\frac{s}{2}} \mathrm{d}x'\, \delta\rho_s(x)f(|x-x'|) = \int_{|x'|\leqslant\frac{s}{2}} \mathrm{d}x'\, \rho_0(x')f(|x-x'|)\,. \tag{37}$$

The uniform level density in (36) and (37) still depends on $V(s)$. However, in the region of interest (19), $\rho_0$ is a constant. One can always choose this constant to be equal to 1 implying that a scale of the one-dimensional motion is set by requiring the average distance between the particles to equal 1. Thus, the integral equation (37) becomes independent of $V$.

It is convenient to introduce new variables

$$z = \frac{s}{2x} \qquad t = \frac{s}{2x'} \qquad u_{\mathrm{g}}(z) = |z|^{-2+\gamma}\delta\rho_s\!\left(\frac{s}{2z}\right) \tag{38}$$

thus reducing (37) to the following one:

$$\int_{-1}^{1} \mathrm{d}t\, \frac{u_{\mathrm{g}}(t)}{|t-z|^{\gamma}} = -v_{\mathrm{g}}(z) \qquad z \in [-1;1]\,. \tag{39}$$

This is exactly the same as the integral equation (15) for the homogeneous density in the variables of (14). Besides the difference in the definitions of the variables for these two cases (38) and (14), the integral equation (39) differs by a specific 'potential', $v_g(z)$, that is obtained by substituting the effective interaction (22) into the integral in the RHS of (37):

$$v_g(z) = \frac{1}{1-\gamma}\frac{(1-|z|)^{1-\gamma}-(1+|z|)^{1-\gamma}}{|z|}. \tag{40}$$

The formal solution to (39) is given by (A16) (one may use it as $v_g(z)$ is an even function of $z$) with $v(\tau)$ being replaced by the explicit expression (40). The appropriate integral has been calculated in appendix B as

$$u_g(z) = \frac{1}{\pi}\cos\left(\frac{\pi\gamma}{2}\right) B\left(\frac{1}{2},\frac{3-\gamma}{2}\right)(1-z^2)^{-\frac{1-\gamma}{2}} F\left(1,\tfrac{1}{2};2-\tfrac{1}{2}\gamma;z^2\right) \tag{41}$$

where $F(\alpha,\beta;\gamma;z)$ is the Gauss hypergeometric function. Changing to the original variables (38), one can immediately obtain $\delta\rho_s$, the density distribution with the gap of width $s$. In the limit $\gamma = 0$, the hypergeometric function becomes elementary, and this solution coincides with the known solution for the Wigner–Dyson problem:

$$\delta\rho_s(x) = \frac{s^2}{8|x|}\frac{F\left(1,\frac{1}{2};\,2;\,\frac{s^2}{4x^2}\right)}{\sqrt{x^2-s^2/4}} = \frac{|x|}{\sqrt{x^2-s^2/4}} - 1. \tag{42}$$

In the case $0<\gamma<1$, we recover the asymptotic behaviour at large $x \gg s/2$:

$$\delta\rho_s(x) = \frac{1}{\pi}\cos\left(\frac{\pi\gamma}{2}\right)B\left(\frac{1}{2},\frac{3-\gamma}{2}\right)\left(\frac{s}{2|x|}\right)^{2-\gamma}. \tag{43}$$

For $|x| \to s/2$, the function $\delta\rho_s(x)$ has the following leading singularity:

$$\delta\rho_s(x) = \frac{1}{\pi}\cos\left(\frac{\pi\gamma}{2}\right) B\left(\frac{1}{2},\frac{1-\gamma}{2}\right)\left(1-\frac{s^2}{4x^2}\right)^{-(1-\gamma)/2}. \tag{44}$$

Using the solution (41), we can calculate the free energy (36) in the presence of the gap. It can be rewritten in the variables (38) as follows:

$$F_s - F_0 = -\frac{A(s/2)^{2-\gamma}}{2}\left\{\int_{-1}^{1}\mathrm{d}z\int_{-1}^{1}\mathrm{d}t\,\frac{u_g(z)\,u_g(t)}{|t-z|^\gamma} - \int_{|z|\leqslant 1}\mathrm{d}z\int_{|t|\leqslant 1}\frac{\mathrm{d}t}{|t-z|^\gamma}\right\}. \tag{45}$$

Here $A$ is the numerical coefficient in the effective interaction (22). Using equation (39), this can be represented as

$$F_s - F_0 = A(s/2)^{2-\gamma}\left\{\int_0^1 \mathrm{d}z\,u_g(z)\,v_g(z) + \frac{2^{2-\gamma}}{(1-\gamma)(2-\gamma)}\right\} \equiv h_\gamma s^{2-\gamma}. \tag{46}$$

The integral in the curly brackets is just a number that defines the coefficient $h_\gamma$. Now, using equation (28) we obtain the asymptotic behaviour of the nearest-level distribution as

$$P(s) \sim \exp\left(-\beta h_\gamma s^{2-\gamma}\right) \qquad s \gg 1. \tag{47}$$

For the standard logarithmic interaction (6), one calculates the integral in (46) for $\gamma = 0$ and, using in this case $c_{d\beta} = 1/\pi^2$ as explained after (23), arrives at the known [3] asymptotic expression for $P(s)$

$$P(s) \sim \exp\left(-\tfrac{1}{16}\pi^2\beta s^2\right) \qquad s \gg 1. \tag{48}$$

Note that the plasma model with an *arbitrary* confinement potential used here gives the exact value of the numerical factor in the exponent, in contrast to the Wigner surmise (27).

Equations (46), (22) and (2) provide a relationship between the coefficients $h_\gamma$ and $c_{d\beta}$ that describe the large-$s$ asymptotics of the two different statistics: the nearest level distribution function, $P(s)$, and the two-level correlation function, $R(s)$. This relationship is convenient to represent as follows:

$$c_{d\beta}\, h_\gamma = \tfrac{1}{16} H_\gamma \qquad (49)$$

where the function $H_\gamma$ is completely determined by (22) and (46) and is chosen so that $H_0 = 1$. This function is computed numerically and represented in figure 1. Although $c_{d\beta}$ could not be calculated analytically [16], equation (49) suggests that independent numerical simulations for the two different statistics can be used not only for verifying the existence of the non-trivial asymptotic behaviour, equations (2) and (47), but also for checking whether the numerical coefficients obey the relationship (49).

## 6. Discussion

The large-energy asymptotic behaviour of the nearest-level distribution, equation (47) is, at the moment, the main physical prediction of the effective plasma model. After this prediction had been published [19], several groups performed the numerical simulations for the level statistics around the mobility edge [24–28] which, in particular, have been used to check this prediction as well as that for the large-energy behaviour of the two-level correlation function (2). It is our aim to show how the results of the detailed analytical calculations presented here can be compared directly with the numerical results.

The two coefficients in the left-hand side of (49) are related to one of the two statistics studied numerically. The amplitude, $c_{d\beta}$, of the two-level correlation function is easy to extract from the level number variance, $\mathrm{var}(N)$, in energy windows containing $N$ levels on average. Namely, one may use that

$$R(N) = -\frac{c_{d\beta}}{\beta N^{2-\gamma}} = \frac{1}{2}\frac{\mathrm{d}^2(\mathrm{var}\, N)}{\mathrm{d}N^2}\,. \qquad (50)$$

The coefficient $h_\gamma$ enters the asymptotic expression for $P(s)$ directly (equation (47)). However, as numerical data for large $s$ can not be very reliable (due to strong fluctuations caused by a small number of realizations available), it is better to use some reasonable interpolation formula that describes $P(s)$ for all $s$. Such an interpolation formula can be constructed by analogy to the Wigner surmise (27), which is approximately valid for the level statistics in the ergodic regime (or, equivalently, for the plasma model with the logarithmic interaction, $\gamma = 0$). The reason, as we have already noted above, is that the problem of level repulsion at very small $s$ is reduced to that of the two levels only. Thus, one should expect for small $s$ that in any regime $P(s) \sim s^\beta$. Combining this with the asymptotic result of (47), one naturally comes to the following surmise [19]:

$$P(s) = Cs^\beta \exp(-\widetilde{h}_\gamma s^{2-\gamma})\,. \qquad (51)$$

The coefficient $\widetilde{h}_\gamma$ in this surmise is not exactly the same as $h_\gamma$ in (47). Both $C$ and $\widetilde{h}_\gamma$ are totally determined from the two normalization conditions, $\int P(s)\,\mathrm{d}s = \int sP(s)\,\mathrm{d}s = 1$ that fix the total probability and the mean-level spacing. For the orthogonal ensemble, $\beta = 1$, it gives

$$\widetilde{h}_\gamma = \left[\frac{\Gamma\left(\frac{3}{2-\gamma}\right)}{\Gamma\left(\frac{2}{2-\gamma}\right)}\right]^{2-\gamma} \qquad C = (2-\gamma)\frac{\Gamma^2\left(\frac{3}{2-\gamma}\right)}{\Gamma^3\left(\frac{2}{2-\gamma}\right)}\,. \qquad (52)$$

For the ergodic regime, $\gamma = 0$, it leads to $\widetilde{h}_0 = \pi/4$ in the Wigner surmise (27), which is different from the exact value $h_0 = \pi^2/16$, equation (48). However, such a 20% difference may be neglected when comparing the analytical results with numeric simulations that have the same accuracy, so that one can identify $\widetilde{h}_\gamma \approx h_\gamma$. Then both expressions (50) and (51) which can be used for direct analysis of numerical data depend on the parameter $\gamma$ only. This parameter is directly related to the critical exponent of the correlation length $\nu$ (equation (2*b*)).

For the 3D Anderson model with the time-reversal symmetry (orthogonal ensemble, $\beta = 1$), the correlation length exponent has been found in numerical simulations [29, 25] to be in the range $\nu \approx 1.3$–$1.6$. Then equation (2*b*) gives the appropriate value of $\gamma$ to be in the range 0.74–0.79. For such values of $\gamma$, the function $H_\gamma$ (figure 1) changes between 0.78 and 0.65. Thus we conclude from (49) that $c_{31}h_\gamma \approx 0.03$–$0.05$. Using the surmise (51), we find from (52) that for the same values of $\gamma$, $h_\gamma \approx 1.5$–$1.3$ so that $c_{31} \approx 0.03$–$0.04$. Now (50) and (51), with the above values of all the relevant parameters, suggest a parameter-free fitting for the data of numeric simulations. When such a fitting has been used, an excellent agreement of the numeric data [30] and the present analytical results has been found. The value of $c_{31}$ found from the numeric data on the number variance [27, 28] or from direct simulation of the correlation function [28], and value of $h_\gamma$ found in [24, 25] are also in good agreement with our predictions.

It is worth noting that $c_{d\beta}$ turns out to be very small. Within the effective plasma model, this results from the fact that the effective interaction which is a continuous function of $\gamma$ (22), goes over to the Wigner–Dyson logarithmic interaction in the limit $\gamma \to 0$ (equation (23)). In the latter case, the exact solution gives $c_{d\beta} = 1/\pi^2 \approx 0.1$. For the critical two-level statistics, an additional small parameter $(1-\gamma) = 0.2$–$0.3$ arises because $H_\gamma$ (figure 1) vanishes linearly near $\gamma = 1$, and the constant $h_\gamma \sim \widetilde{h}_\gamma$ remains finite in this limit. In the limit $\gamma = 1$, the effective interaction disappears, and one comes naturally to the Poisson asymptotic behaviour in the surmise (51). Such a surmise with $\gamma = 1$ was first suggested phenomenologically by Shklovskii *et al* [15]. Here we have shown, using the analytical results for the level correlations at the mobility edge [16] and the mapping onto the effective plasma model, that the non-trivial asymptotic behaviour of the nearest-level distribution is just due to the presence of the effective interaction between the levels.

In conclusion, we note that in the above mapping we considered only a pairwise interaction $f(s_i - s_j)$ between 'particles' of the effective plasma model. This was sufficient to restore the microscopically known two-level correlation function. In general, no factorization of many-level correlation functions is expected to happen in a non-ergodic regime (in contrast to the ergodic regime of RMT), so that there could be many-body terms

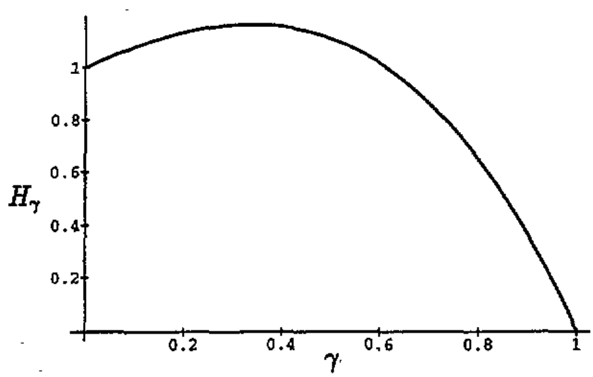


**Figure 1.** The function $H_\gamma$ (equation (49)), which relates the numerical coefficients relevant for the two different statistics.

in the effective interaction. Such terms arise, for instance, in the generalized random matrix ensemble considered recently in [32]. However, for physically reasonable models, they are unlikely to be relevant for large-distance asymptotic properties of different statistics, such as the nearest-level distribution function $P(s)$ at large $s$. The reason is that the many-level correlation functions are obviously very small when distances between each pair of levels are large. Their main contribution is expected when only one such a difference is large and all the rest are of order $\Delta$. Then the impact of the higher correlations is reduced to the renormalization of the constant $c_{d\beta}$ in the effective pairwise interaction. This constant is unknown anyway and may only be estimated numerically as described above.

Therefore, whatever the exact joint probability distribution of the energy levels is, the mapping described in the present paper seems to be a very reasonable phenomenological framework for studying long-range asymptotic properties of different spectral statistics near the Anderson transition.

## Acknowledgments

We gratefully acknowledge a contribution at an earlier stage of this work from Arkadii Aronov, now sadly deceased. We are thankful to Y Gefen, J M F Gunn, J T Chalker, Y Chen, B Shapiro and B Shklovskii for useful discussions on this and related subjects. One of the authors (IVL) is thankful to the Institute of Theoretical Physics (Technion, Haifa) and to the Weizmann Institute of Science (Rehovot) for kind hospitality extended to him at an earlier stage of this work. Work in Birmingham has been supported by EPSRC grant GR/J35238.

## Appendix A. Solution of the integral equation

The idea of the solution is to represent the singular kernel of (15) which does not have resolvent as a product of two Volterra operators, each resolvable [23]. We introduce the operator notation

$$(P_\gamma u)(z) = -v(z) \qquad (P_\gamma u)(z) \equiv \int_{-1}^{1} \mathrm{d}t \, \frac{1}{|t-z|^\gamma} u(t) \tag{A1}$$

and look for the operator representation

$$P_\gamma \equiv C^{-1} K_\gamma K_\gamma^\dagger \, . \tag{A2}$$

Such a representation allows to rewrite (A1) as the following set of two equations:

$$(K_\gamma \phi)(z) = v(z) \qquad (K_\gamma^\dagger u)(z) = -C\phi(z) \, . \tag{A3}$$

Here $C$ is some positive constant, and $K_\gamma$ and $K_\gamma^\dagger$ are conjugate Volterra operators defined as

$$(K_\gamma u)(z) \equiv \int_{-1}^{z} \mathrm{d}t \, k(z;t) u(t) (K_\gamma^\dagger u)(z) \equiv \int_{z}^{1} \mathrm{d}t \, k(t;z) u(t) \, . \tag{A4}$$

Notations are justified by the fact that $((K_\gamma u, \phi)) = ((u, K_\gamma^\dagger \phi))$ where $((u, \phi)) = \int_{-1}^{1} u\phi \, \mathrm{d}t$. For a positive operator $P$, the representation of the form (A2) always exists [23]. Then, having found the resolvents $R_\gamma$ and $R_\gamma^\dagger$, for the operators $K_\gamma$ and $K_\gamma^\dagger$, one finds the solution to (15) as

$$u(z) = -C \, (R_\gamma^\dagger R_\gamma v)(z) \, . \tag{A5}$$

The first step is to find the operators $K_\gamma$ and $K_\gamma^\dagger$ explicitly. Substituting the definitions (A4) into the operator relation (A2), and changing the order of integration in the following convergent integral, one has

$$(K_\gamma K_\gamma^\dagger u)(z) = \int_{-1}^{z} \mathrm{d}\tau\, k_\gamma(z;\tau) \int_{\tau}^{1} \mathrm{d}t\, k_\gamma(t;\tau) u(t)$$
$$= \int_{-1}^{1} \mathrm{d}t \left\{ \int_{-1}^{\min(z,t)} \mathrm{d}\tau\, k_\gamma(z;\tau) k_\gamma(t;\tau) \right\} u(t)\,. \tag{A6}$$

Therefore, the factorization (A2) requires the kernel of the operator (A4) to obey the following identity:

$$\frac{1}{|z-t|^\gamma} = C^{-1} \int_{-1}^{\min(z,t)} \mathrm{d}\tau\, k_\gamma(z;\tau) k_\gamma(t;\tau)\,. \tag{A7}$$

Such a kernel is given by

$$k_\gamma(z;\tau) = \left(\frac{\tau+1}{z+1}\right)^{\frac{\gamma-1}{2}} (z-\tau)^{-\frac{1+\gamma}{2}}\,. \tag{A8}$$

To prove it, one introduces in (A7) a new variable of integration,

$$w = \frac{\tau+1}{a(z-\tau)} \qquad \Rightarrow \qquad \tau = \frac{aw(z+1)}{1+aw} - 1$$

where the constant, $a$, equals to $(t+1)(z-t)^{-1}$. Then the integral in (A7) is reduced for $z > t$ to

$$[(z+1)(t+1)]^{\frac{1-\gamma}{2}} \int_{-1}^{z} \mathrm{d}\tau\, \frac{[(z-\tau)(t-\tau)]^{-\frac{1+\gamma}{2}}}{(\tau+1)^{1-\gamma}} = \frac{1}{(z-t)^\gamma} \int_0^1 \mathrm{d}w\, w^{\gamma-1}(1-w)^{-\frac{1+\gamma}{2}}\,. \tag{A9}$$

Using the definition of the Euler B-function [33],

$$\mathrm{B}(\alpha,\beta) = \int_0^1 \mathrm{d}u\, u^{\alpha-1}(1-u)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{A10}$$

one finds that with the constant $C$ chosen as

$$C = \mathrm{B}\left(\gamma, \frac{1-\gamma}{2}\right) \tag{A11}$$

the identity (A7) holds for $z > t$. Similarly, one may verify this identity for $z < t$.

The second step is to construct the resolvent of $K_\gamma$ and $K_\gamma^\dagger$. To this end, we define

$$(\widetilde{K}_\gamma \phi)(z) = \int_{-1}^{z} \mathrm{d}t\ (t+1)^{\frac{\gamma-1}{2}} (z-t)^{\frac{\gamma-1}{2}} \phi(t)\,. \tag{A12}$$

Then, acting by this operator on $K\phi$ and changing the order of integrations, we have

$$(\widetilde{K}_\gamma K_\gamma \phi)(z) = \int_{-1}^{z} \mathrm{d}t\, (t+1)^{\frac{\gamma-1}{2}} (z-t)^{\frac{\gamma-1}{2}} \int_{-1}^{t} \mathrm{d}\tau \left(\frac{t+1}{\tau+1}\right)^{\frac{1-\gamma}{2}} (t-\tau)^{-\frac{1+\gamma}{2}} \phi(\tau)$$
$$= \int_{-1}^{z} \mathrm{d}\tau (\tau+1)^{\frac{\gamma-1}{2}} \phi(\tau) \int_{\tau}^{z} \mathrm{d}t (z-t)^{\frac{\gamma-1}{2}} (t-\tau)^{-\frac{1+\gamma}{2}}$$
$$= \pi \sec\left(\frac{\pi\gamma}{2}\right) \int_{-1}^{z} \mathrm{d}\tau (\tau+1)^{\frac{\gamma-1}{2}} \phi(\tau)$$

where we used (A10), and $B(\frac{1+\gamma}{2}, \frac{1-\gamma}{2}) = \pi \sec\left(\frac{\pi\gamma}{2}\right)$. Thus, equation (A3) is reduced to

$$\int_{-1}^{z} d\tau (\tau+1)^{\frac{\gamma-1}{2}} \phi(\tau) = \frac{\cos(\pi\gamma/2)}{\pi} \int_{-1}^{z} dt\ (t+1)^{\frac{\gamma-1}{2}} (z-t)^{\frac{\gamma-1}{2}} v(t) \tag{A13}$$

whose solution is immediately found by differentiating with respect to $z$ as

$$\phi(z) = (R_\gamma v)(z) \equiv \frac{\cos(\pi\gamma/2)(z+1)^{\frac{1-\gamma}{2}}}{\pi} \frac{d}{dz} \int_{-1}^{z} d\tau\ (\tau+1)^{\frac{\gamma-1}{2}} (z-\tau)^{\frac{\gamma-1}{2}} v(\tau). \tag{A14}$$

In a similar way, one solves the second integral equation in (A3)

$$u(t) = -C\,(R_\gamma^\dagger \phi)(t) \equiv C \frac{\cos(\pi\gamma/2)(t+1)^{\frac{\gamma-1}{2}}}{\pi} \frac{d}{dt} \int_{t}^{1} dz\ (z+1)^{\frac{1-\gamma}{2}} (z-t)^{\frac{\gamma-1}{2}} \phi(z). \tag{A15}$$

Now, substituting the expressions for the resolvents (equations (A14) and (A15)) and the constant $C$ (equation (A11)), into the formal solution (A4) to the integral equation and changing to the original variables (14), we obtain the explicit solution (16), given in the text.

If $v(\tau) = v(-\tau)$, the solution to (15) may be represented in the following form which is more convenient for performing the integration:

$$u(t) = \frac{2^{2-\gamma}\cos^2\left(\frac{\pi\gamma}{2}\right)}{\pi^2} B\left(\gamma, \frac{1-\gamma}{2}\right) \frac{d}{d(t^2)} \left\{ \int_{t}^{1} dz\, (z^2-t^2)^{\frac{\gamma-1}{2}} z^{2-\gamma} \times \frac{d}{dz}\left[ \int_{0}^{z} d\tau\, (z^2-\tau^2)^{\frac{\gamma-1}{2}} v(\tau) \right] \right\}. \tag{A16}$$

After changing to the original variables (14), we arrive at the explicit solution in the form of (25).

## Appendix B. Calculation of the integral $u_n$

To calculate the integral (A16) with the 'potential' $v_g(\tau)$ given by (40), it is convenient to expand $v_g(\tau)$ in the powers of $\tau^2$. Thus we obtained the solution in the following form:

$$u_g(t) = \frac{2}{\gamma-1} \sum_{n=0}^{\infty} \binom{1-\gamma}{2n+1} u_n(t). \tag{B1}$$

Here $\binom{a}{b}$ denotes the binomial coefficient, and $u_n(t)$ is the integral (A16) obtained by substituting $v(\tau) = \tau^{2n}$. After performing the integration over $\tau$ in this integral with the help of (A10), one reduces it to

$$u_n(t) = 2A_n \frac{d}{d(t^2)} \int_{t}^{1} dz\, (z^2-t^2)^{-\alpha} z^{2n+1} = A_n \frac{d}{d(t^2)} \int_{t^2}^{1} dz\, (z-t^2)^{-\alpha} z^n \tag{B2}$$

$$A_n \equiv \frac{2^{2\alpha}\sin^2(\pi\alpha)}{\pi^2} \left(n-\alpha+\tfrac{1}{2}\right) B\left(n+\tfrac{1}{2}, 1-\alpha\right) B(1-2\alpha, \alpha) \qquad \alpha \equiv \frac{1-\gamma}{2}. \tag{B3}$$

By expanding the binom, $z^n = [1-(1-z)]^n$, and using (A10), one obtains the following result:

$$u_n(t) = A_n \sum_{k=0}^{n} \binom{n}{k} (1-t^2)^{n-k-\alpha} (-1)^{n-k+1} (n-k-\alpha+1) B(n-k+1, 1-\alpha). \tag{B4}$$

Finally, using the properties of the Euler functions, one arrives at

$$u_n(t) = \frac{(2n+1-2\alpha)\sin(\pi\alpha)}{\pi} \mathrm{B}(2n+1, 1-2\alpha) \sum_{k=0}^{n} \binom{n-\alpha}{k} (-1)^{n-k+1}(1-t^2)^{n-\alpha-k}. \tag{B5}$$

On substituting this result into (B1), we further simplify the coefficients to obtain

$$u_g(t) = \frac{2\cos\left(\frac{\pi\gamma}{2}\right)}{\pi}(1-t^2)^{-\frac{1-\gamma}{2}} I(t) \tag{B6}$$

$$I(t) = \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{k=0}^{n} (-1)^k \binom{n-\frac{1-\gamma}{2}}{n-k} (1-t^2)^k. \tag{B7}$$

This can be transformed by expressing the factor $(2n+1)^{-1}$ as $\int_0^1 \xi^{2n}\, d\xi$ and changing the order of summations and integration so that the double sum in (B6) is represented as

$$\begin{aligned} I(t) &= \int_0^1 d\xi \sum_{k=0}^{\infty} (-1)^k \left[(1-t^2)\xi^2\right]^k \sum_{n=0}^{\infty} \binom{k+n-\frac{1-\gamma}{2}}{n} \xi^{2n} \\ &= \int_0^1 d\xi\, (1-\xi^2)^{-\frac{1+\gamma}{2}} \sum_{k=0}^{\infty} (-1)^k \left[\frac{(1-t^2)\xi^2}{1-\xi^2}\right]^k = \int_0^1 d\xi \frac{(1-\xi^2)^{\frac{1-\gamma}{2}}}{1-\xi^2 t^2}. \end{aligned} \tag{B8}$$

Introducing a new variable of the integration, $w = \xi^2$, we recognize in the last integral a standard representation [33] of the Gauss hypergeometric function, $F$, so that

$$I(t) = \tfrac{1}{2}\mathrm{B}\left(\frac{1}{2}, \frac{3-\gamma}{2}\right) F\left(1, \tfrac{1}{2}; 2-\tfrac{1}{2}\gamma; t^2\right). \tag{B9}$$

Combining equations (B6)–(B9), we obtain (41) given in the text.

## References


[1] Wigner E P 1951 *Proc. Camb. Philos. Soc.* **47** 790
[2] Dyson F J 1962 *J. Math. Phys.* **3** 140
[3] Mehta M L 1991 *Random Matrices* (Boston, MA: Academic)
Haake F 1991 *Quantum Signatures of Chaos* (Berlin: Springer)
[4] Gor'kov L P and Eliashberg G M 1965 *Sov. Phys.–JETP* **21** 940
[5] Abrikosov A A, Gor'kov L P and Dzyaloshinskii I E 1965 *Methods of Quantum Field Theory in Statistical Physics* (New York: Pergamon)
Gor'kov L P, Larkin A I and Khmel'nitskii D E 1979 *JETP Lett.* **30** 228
Hikami S 1981 *Phys. Rev.* B **24** 2671
[6] Schäfer L and Wegner F 1980 *Z. Phys.* B **38** 113
Efetov K B, Larkin A I and Khmel'nitskii D E 1980 *Sov. Phys.–JETP* **52** 568
[7] Efetov K B 1983 *Adv. Phys.* **32** 53
[8] Altshuler B L and Shklovskii B I 1986 *Sov. Phys.–JETP* **64** 127
[9] Altshuler B L and Szafer A 1993 *Phys. Rev. Lett.* **70** 5879
Simons B D and Altshuler B L 1993 *Phys. Rev. Lett.* **70** 4063
Simons B D, Altshuler B L and Lee P A 1994 *Phys. Rev. Lett.* **72** 64
[10] Gutzwiller M C 1971 *J. Math. Phys.* **12** 343
[11] Berry M V 1985 *Proc. R. Soc.* A **400** 229
[12] Argaman N, Imry Y and Smilansky U 1993 *Phys. Rev.* B **47** 4440
[13] Thouless D C 1974 *Phys. Rep.* C **13** 93
[14] Lee P A and Ramakrishnan T V 1985 *Rev. Mod. Phys.* **57** 287
[15] Shklovskii B I, Shapiro B, Sears B R, Lambrianides P and Shore H B 1993 *Phys. Rev.* B **47** 11487
[16] Kravtsov V E, Lerner I V, Altshuler B L and Aronov A G 1994 *Phys. Rev. Lett.* **72** 888
Aronov A G, Kravtsov V E and Lerner I V 1995 *Phys. Rev. Lett.* **74** 1174

[17] Dyson F J 1972 *J. Math. Phys.* **13** 90
[18] Jalabert R A, Pichard J L and Beenakker C W J 1993 *Europhys. Lett.* **24** 1
[19] Aronov A G, Kravtsov V E and Lerner I V 1994 *JETP Lett.* **59** 40
[20] Beenakker C W J 1993 *Phys. Rev. Lett.* **70** 1155
[21] Brézin E and Zee A 1993 *Nucl. Phys.* B **402** 613
Weidenmüller H A and Hackenbroich G 1995 Universality of random-matrix results for non-Gaussian ensembles (unpublished)
[22] Muttalib K A, Chen Y, Ismail M E H and Nicopoulos V N 1993 *Phys. Rev. Lett.* **71** 471
Blecken C, Chen Y and Muttalib K A 1994 *J. Phys. A: Math. Gen.* **27** L563
Canali C M, Wallin M and Kravtsov V E 1995 *Phys. Rev.* B **51** 2831
Canali C M and Kravtsov V E 1995 *Phys. Rev.* E **51**
[23] Porter D and Stirling D S G 1990 *Integral Equations* (Cambridge: Cambridge University Press)
[24] Aronov A G and Mirlin A D 1995 *Phys. Rev.* B **51** 6131
Kravtsov V E and Lerner I V 1995 *Phys. Rev. Lett.* **74** 2563
[25] Evangelou S N 1994 *Phys. Rev.* B **49** 16805
Evangelou S N 1994 *Prog. Theor. Phys.* **116** 319
[26] Hofstetter E and Schreiber M 1994 *Phys. Rev. Lett.* **73** 3137
Varga I, Hofstetter E, Schreiber M and Pipek J 1994 Shape analysis of the level spacing distribution around the metal insulator transition in the three dimensional Anderson model *Preprint* cond-mat/9407058
[27] Sears B R and Shore H B 1994 Statistical fluctuations of energy level spectra near the metal–insulator transition (unpublished)
[28] Braun D and Montambaux G 1995 Spectral correlations at the metal–insulator transition (unpublished)
[29] Zharekeshev I and Kramer B 1995 Scaling of level statistics at the disorder induced metal–insulator transition (unpublished)
[30] Ohtsuki T and Ono Y 1995 Private communication
[31] MacKinnon A and Kramer B 1993 *Rep. Prog. Phys.* **56** 1469
[32] Moshe M, Neuberger H and Shapiro B 1994 *Phys. Rev. Lett.* **73** 1497
[33] Gradstein I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (Boston, MA: Academic)